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Exact Green's function of the Aharonov–Bohm–Coulomb system via the Feynman–Kac formula

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Abstract. The Green's function of the relativistic Aharonov–Bohm–Coulomb system is given by the Feynman–Kac formula. The earlier treatment is based on the multiple-valued transformation of *Levi-Civitá*. The method used in this contribution involves only the explicit form of a simple Green's function and an explicit path integral is avoided.

1. Introduction

In recent years, the Aharonov–Bohm (AB) effect [1] has been of much interest in the context of anyonic theories [2]. Since an anyon is a two-dimensional object that carries the magnetic flux, the dominant interaction between anyons is the AB interaction. An anyon–charge interaction naturally requires the Coulomb modification. The present paper deals with the relativistic Aharonov–Bohm–Coulomb (ABC) system and its aim is to derive the Green's function by making use of the Feynman–Kac formula. So clear and neat is the method that it provides us not only with an alternative approach but a completely diverse viewpoint for treating physical problems. The ABC case can serves as a prototype for the treatment of Besselian-type problems [3] via the Feynman–Kac formula.

2. Green's function of the ABC system

The starting point is the path-integral representation for the Green's function of a relativistic particle in external electromagnetic fields [4, 5]:

$$G(\boldsymbol{x}_b, \boldsymbol{x}_a; E) = \frac{\mathrm{i}\hbar}{2mc} \int_0^\infty \mathrm{d}S \int \mathcal{D}\rho(\lambda)\, \Phi[\rho(\lambda)] \int \mathcal{D}^D x\, \mathrm{e}^{-A_E[\boldsymbol{x},\dot{\boldsymbol{x}}]/\hbar}. \tag{2.1}$$

The action integral

$$A_E[\boldsymbol{x}, \dot{\boldsymbol{x}}] = \int_{\lambda_a}^{\lambda_b} \mathrm{d}\lambda \left[\frac{m}{2\rho(\lambda)} \dot{\boldsymbol{x}}^2(\lambda) - \mathrm{i}\frac{e}{c} \boldsymbol{A}(\boldsymbol{x}) \cdot \dot{\boldsymbol{x}}(\lambda) - \rho(\lambda) \frac{(E - V(\boldsymbol{x}))^2}{2mc^2} + \rho(\lambda) \frac{mc^2}{2} \right] \tag{2.2}$$

where S is defined by

$$S = \int_{\lambda_a}^{\lambda_b} \mathrm{d}\lambda\, \rho(\lambda) \tag{2.3}$$

in which $\rho(\lambda)$ is an arbitrary dimensionless fluctuating scale variable, and $\Phi[\rho(\lambda)]$ is some convenient gauge-fixing functional, such as $\Phi[\rho] = \delta[\rho - 1]$, to fix the value of $\rho(\lambda)$ to unity [5, 6]. $\hbar/mc$ is the well known Compton wavelength of a particle of mass m, $\boldsymbol{A}(\boldsymbol{x})$ and $V(\boldsymbol{x})$ denote the vector and scalar potential of the systems, respectively. E is the system energy and $\boldsymbol{x}$ is the spatial part of the $(D+1)$ vector $x = (\boldsymbol{x}, \tau)$.

The functional integral for $\boldsymbol{x}$ in the representation (2.1) can be interpreted as the expectation value of the real functional $\exp\{-(1/\hbar)\int_{\lambda_a}^{\lambda_b} d\lambda\, \beta\rho(\lambda)V(\boldsymbol{x}(\lambda))\}$ over the measure

$$K_0(\boldsymbol{x}_b, \boldsymbol{x}_a; \lambda_b - \lambda_a) = \int \mathcal{D}^D x(\lambda) \times \exp\left\{-\frac{1}{\hbar}\int_{\lambda_a}^{\lambda_b} d\lambda \left[\frac{m}{2\rho(\lambda)}\dot{\boldsymbol{x}}^2(\lambda) - i\frac{e}{c}\boldsymbol{A}(x)\cdot\dot{\boldsymbol{x}}(\lambda) - \rho(\lambda)\frac{V(\boldsymbol{x})^2}{2mc^2}\right]\right\} \qquad (2.4)$$

and the entire Green's function reduces to the following formula:

$$G(\boldsymbol{x}_b, \boldsymbol{x}_a; E) = \frac{i\hbar}{2mc}\int_0^\infty dS \int \mathcal{D}\rho(\lambda)\Phi[\rho(\lambda)] \times \exp\left\{-\frac{1}{\hbar}\int_{\lambda_a}^{\lambda_b} d\lambda\, \rho(\lambda)\,\mathcal{E}\right\}\left\langle \exp\left\{-\frac{1}{\hbar}\int_{\lambda_a}^{\lambda_b} d\lambda\, \beta\rho(\lambda)V(\boldsymbol{x}(\lambda))\right\}\right\rangle \qquad (2.5)$$

in which $\mathcal{E} = (m^2c^4 - E^2)/2mc^2$, $\beta = E/mc^2$ with the notation $\langle\star\rangle$ denoting the expectation value of the moment $\star$ over the measure $K_0(\boldsymbol{x}_b, \boldsymbol{x}_a; \lambda_b - \lambda_a)$. Equation (2.5) forms the basis for studying the relativistic potential problems by the Feynman–Kac-type formula.

Expanding the potential $V(\boldsymbol{x})$ in equation (2.5) into a power series and interchanging the order of integration and summation, we have

$$G(\boldsymbol{x}_b, \boldsymbol{x}_a; E) = \frac{i\hbar}{2mc}\int_0^\infty dS \int \mathcal{D}\rho\, \Phi[\rho]\exp\left\{-\frac{1}{\hbar}\int_{\lambda_a}^{\lambda_b} d\lambda\, \rho(\lambda)\mathcal{E}\right\} \times \sum_{n=0}^{\infty}\frac{(-\beta/\hbar)^n}{n!}\left\langle\left(\int_{\lambda_a}^{\lambda_b} d\lambda\, \rho(\lambda)\, V(\boldsymbol{x}(\lambda))\right)^n\right\rangle. \qquad (2.6)$$

Ordering λ as $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \lambda_b$ and denoting $\boldsymbol{x}(\lambda_i) = \boldsymbol{x}_i$, the perturbation series in equation (2.6) explicitly turns into [7]

$$\sum_{n=0}^{\infty}\frac{(-\beta/\hbar)^n}{n!}\left\langle\left(\int_{\lambda_a}^{\lambda_b} d\lambda\, \rho(\lambda)V(\boldsymbol{x}(\lambda))\right)^n\right\rangle = K_0(\boldsymbol{x}_b, \boldsymbol{x}_a; \lambda_b - \lambda_a) + \sum_{n=1}^{\infty}\left(-\frac{\beta}{\hbar}\right)^n\int_{\lambda_a}^{\lambda_b} d\lambda_n \int_{\lambda_a}^{\lambda_n} d\lambda_{n-1}\cdots\int_{\lambda_a}^{\lambda_2} d\lambda_1 \times \int\left[\prod_{j=0}^{n}K_0(\boldsymbol{x}_{j+1}, \boldsymbol{x}_j; \lambda_{j+1} - \lambda_j)\right]\prod_{i=1}^{n}\rho_i V(\boldsymbol{x}_i)\, d\boldsymbol{x}_i \qquad (2.7)$$

where $\lambda_0 = \lambda_a$, $\lambda_{n+1} = \lambda_b$, $\boldsymbol{x}_{n+1} = \boldsymbol{x}_b$ and $\boldsymbol{x}_0 = \boldsymbol{x}_a$. In the case of the ABC potential system in two dimensions, we have

$$\boldsymbol{A}(\boldsymbol{x}) = 2g\frac{-x_2\hat{e}_1 + x_1\hat{e}_2}{x_1^2 + x_2^2} \qquad V(r) = -\frac{e^2}{r} \qquad (2.8)$$

where $\hat{e}_{1,2}$ stands for the unit vector along the x, y axes, respectively. The perturbation expansion in equation (2.7) becomes

$$\sum_{n=1}^{\infty} \frac{(-\beta/\hbar)^n}{n!} \left\langle \left(\int_{\lambda_a}^{\lambda_b} \mathrm{d}\lambda\, \rho(\lambda)\, V(\boldsymbol{x}(\lambda)) \right)^n \right\rangle$$

$$= K_0(\boldsymbol{x}_b, \boldsymbol{x}_a; \lambda_b - \lambda_a) + \sum_{n=1}^{\infty} \left(\frac{\beta e^2}{\hbar} \right)^n \int_{\lambda_a}^{\lambda_b} \mathrm{d}\lambda_n \int_{\lambda_a}^{\lambda_n} \mathrm{d}\lambda_{n-1} \cdots \int_{\lambda_a}^{\lambda_2} \mathrm{d}\lambda_1$$

$$\times \int \left[\prod_{j=0}^{n} K_0(\boldsymbol{x}_{j+1}, \boldsymbol{x}_j; \lambda_{j+1} - \lambda_j) \right] \prod_{i=1}^{n} \rho_i \frac{\mathrm{d}\boldsymbol{x}_i}{r_i} \tag{2.9}$$

with

$$K_0(\boldsymbol{x}_b, \boldsymbol{x}_a; \lambda_b - \lambda_a) = \int \mathcal{D}^2 x \exp\left\{ -\frac{1}{\hbar} \int_{\lambda_a}^{\lambda_b} \mathrm{d}\lambda \right.$$

$$\left. \times \left[\frac{m}{2\rho(\lambda)} \boldsymbol{x}'^2(\lambda) - \mathrm{i}\frac{e}{c} \boldsymbol{A}(x) \cdot \dot{\boldsymbol{x}}(\lambda) - \rho(\lambda) \frac{\hbar^2}{2m} \frac{\alpha^2}{r^2} \right] \right\} \tag{2.10}$$

where $\alpha = e^2/\hbar c$ is the fine structure constant. We now choose $\Phi[\rho] = \delta[\rho - 1]$ to fix the value of $\rho(\lambda)$ to unity. The Green's function in equation (2.7) becomes

$$G(\boldsymbol{x}_b, \boldsymbol{x}_a; E) = \frac{\mathrm{i}\hbar}{2mc} \int_0^{\infty} \mathrm{d}S\, \mathrm{e}^{-(\mathcal{E}/\hbar)S} \left\{ K_0(\boldsymbol{x}_b, \boldsymbol{x}_a; S) \right.$$

$$+ \sum_{n=1}^{\infty} \left(\frac{\beta e^2}{\hbar} \right)^n \int_{\lambda_a}^{\lambda_b} \mathrm{d}\lambda_n \int_{\lambda_a}^{\lambda_n} \mathrm{d}\lambda_{n-1} \cdots \int_{\lambda_a}^{\lambda_2} \mathrm{d}\lambda_1$$

$$\left. \times \int \left[\prod_{j=0}^{n} K_0(\boldsymbol{x}_{j+1}, \boldsymbol{x}_j; \lambda_{j+1} - \lambda_j) \right] \prod_{i=1}^{n} \frac{\mathrm{d}\boldsymbol{x}_i}{r_i} \right\}. \tag{2.11}$$

We observe that the integration over S is a Laplace transformation. Because of the convolution property of the Laplace transformation, we obtain

$$G(\boldsymbol{x}_b, \boldsymbol{x}_a; E) = \frac{\mathrm{i}\hbar}{2mc} \left\{ G_0(\boldsymbol{x}_b, \boldsymbol{x}_a; \mathcal{E}) + \sum_{n=1}^{\infty} \left(\frac{\beta e^2}{\hbar} \right)^n \int \left[\prod_{j=0}^{n} G_0(\boldsymbol{x}_{j+1}, \boldsymbol{x}_j; \mathcal{E}) \right] \prod_{i=1}^{n} \frac{\mathrm{d}\boldsymbol{x}_i}{r_i} \right\}. \tag{2.12}$$

To go further, let us analyse the influence of the AB effect. Introducing the azimuthal angle around the AB tube

$$\varphi(\boldsymbol{x}) = \arctan(x_2/x_1) \tag{2.13}$$

the components of the vector potential can be expressed as

$$A_i = 2g \partial_i \varphi(\boldsymbol{x}). \tag{2.14}$$

The associated magnetic field lines are confined to an infinitely thin tube along the z-axis:

$$B_3 = 2g \epsilon_{3ij} \partial_i \partial_j \varphi(\boldsymbol{x}) = 4\pi g \delta(\boldsymbol{x}_\perp) \tag{2.15}$$

where $\boldsymbol{x}_\perp$ stands for the transverse vector $\boldsymbol{x}_\perp = (x_1, x_2)$. Note that the derivatives in front of $\varphi(\boldsymbol{x})$ commute everywhere, except at the origin where Stokes' theorem yields

$$\int \mathrm{d}^2 x \left(\partial_1 \partial_2 - \partial_2 \partial_1 \right) \varphi(\boldsymbol{x}) = \oint \mathrm{d}\varphi = 2\pi. \tag{2.16}$$

The magnetic flux through the tube is defined by the integral

$$\Omega = \int \mathrm{d}^2x\, B_3. \tag{2.17}$$

This shows that the coupling constant g is related to the magnetic flux by

$$g = \frac{\Omega}{4\pi}. \tag{2.18}$$

Inserting $A_i = 2g\partial_i\varphi(\boldsymbol{x})$ into the action of equation (2.10), the magnetic interaction takes the form

$$\mathcal{A}_{\mathrm{mag}} = -\hbar\beta_0 \int_0^S \mathrm{d}\lambda\, \dot{\varphi}(\lambda) \tag{2.19}$$

where $\varphi(\lambda) = \varphi(\boldsymbol{x}(\lambda))$, $\dot{\varphi} = \mathrm{d}\varphi/\mathrm{d}\lambda$ and β_0 is the dimensionless number

$$\beta_0 = -\frac{2eg}{\hbar c}. \tag{2.20}$$

The minus sign is a matter of convention. Since the particle orbits are present at all times, their worldlines in spacetime can be considered as being closed at infinity, and the integral

$$n = \frac{1}{2\pi} \int_0^S \mathrm{d}\lambda\, \dot{\varphi}(\lambda) \tag{2.21}$$

is the topological invariant with integer values of the winding number n. The magnetic interaction is therefore purely topological, its value being

$$\mathcal{A}_{\mathrm{mag}} = -\hbar\beta_0 2n\pi. \tag{2.22}$$

Added to the action of equation (2.10) in the radial decomposition of the relativistic path integral [3, 6, 8] and expressing the sum over the azimuthal quantum number by Poisson's summation formula (e.g. [8])

$$\sum_{k=-\infty}^{\infty} f(k) = \int_{-\infty}^{\infty} \mathrm{d}y \sum_{n=-\infty}^{\infty} \mathrm{e}^{2\pi n y \mathrm{i}} f(y) \tag{2.23}$$

we obtain

$$G_0(\boldsymbol{x}_b, \boldsymbol{x}_a; E) = \frac{\mathrm{i}\hbar}{2mc} \int_0^\infty \mathrm{d}S\, \mathrm{e}^{-S\mathcal{E}/\hbar} \int_{-\infty}^{\infty} \mathrm{d}z\, g_z(r_b, r_a; S)$$
$$\times \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \exp\left[\mathrm{i}(z - \beta_0)(\varphi_b + 2n\pi - \varphi_a)\right] \tag{2.24}$$

where the pseudopropagator $g_z(r_b, r_a; S)$ reads

$$g_z(r_b, r_a; S) = \frac{m}{\hbar} \frac{1}{S} \mathrm{e}^{-m(r_b^2 + r_a^2)/2\hbar S} I_{\sqrt{z^2-\alpha^2}}\left(\frac{M r_b r_a}{\hbar S}\right). \tag{2.25}$$

The sum over all n in equation (2.24) forces z to be equal to β_0 modulo an arbitrary integral number. The result is

$$G_0(\boldsymbol{x}_b, \boldsymbol{x}_a; E) = \frac{\mathrm{i}\hbar}{2mc} \int_0^\infty \mathrm{d}S\, \mathrm{e}^{-S\mathcal{E}/\hbar} K_0(\boldsymbol{x}_b, \boldsymbol{x}_a; S) \tag{2.26}$$

in which $K(\boldsymbol{x}_b, \boldsymbol{x}_a; S)$ is given by

$$K_0(\boldsymbol{x}_b, \boldsymbol{x}_a; S) = \sum_{k=-\infty}^{\infty} g_{k+\beta_0}(r_b, r_a; S) \frac{1}{2\pi} \mathrm{e}^{\mathrm{i}k(\varphi_b - \varphi_a)}. \tag{2.27}$$

With the help of the representation, we can perform the angular decomposition of equation (2.12). Integrating over the intermediate angular part, we arrive at

$$G(\boldsymbol{x}_b, \boldsymbol{x}_a; E) = \frac{\mathrm{i}\hbar}{2mc} \sum_{k=-\infty}^{\infty} G_k(r_b, r_a; \mathcal{E}) \frac{1}{2\pi} \mathrm{e}^{\mathrm{i}k(\varphi_b - \varphi_a)}. \tag{2.28}$$

The pure radial amplitude $G_k(r_b, r_a; \mathcal{E})$ has the form

$$G_k(r_b, r_a; \mathcal{E}) = \frac{m}{\hbar} \sum_{n=0}^{\infty} \left(\frac{m\beta e^2}{\hbar^2} \right)^n g^{(n)}_{k+\beta_0}(r_b, r_a; \mathcal{E}) \tag{2.29}$$

with $g^{(n)}_{k+\beta_0}$ given by

$$g^{(n)}_{k+\beta_0}(r_b, r_a; \mathcal{E}) = \int_0^\infty \cdots \int_0^\infty \left[\prod_{j=0}^{n} g^{(0)}_{k+\beta_0}(r_{j+1}, r_j; \mathcal{E}) \right] \prod_{i=1}^{n} \mathrm{d}r_i. \tag{2.30}$$

To obtain the explicit result of $g^{(n)}_{k+\beta_0}$, we note that [9]

$$\begin{aligned} &\int_0^\infty \frac{\mathrm{d}S}{S} \mathrm{e}^{-(\mathcal{E}/\hbar)S} \mathrm{e}^{-m(r_b^2+r_a^2)/2\hbar S} I_{\sqrt{|k+\beta_0|^2-\alpha^2}} \left(\frac{m}{\hbar} \frac{r_b r_a}{S} \right) \\ &\qquad = 2 \int_0^\infty \mathrm{d}z \, \frac{1}{\sinh z} \mathrm{e}^{-\kappa(r_b+r_a)\coth z} I_{2\sqrt{|k+\beta_0|^2-\alpha^2}} \left(\frac{2\kappa\sqrt{r_b r_a}}{\sinh z} \right) \end{aligned} \tag{2.31}$$

with $\kappa = \sqrt{m^2c^4 - E^2}/\hbar c$. we have, by using the formula (e.g. [8])

$$\int_0^\infty \mathrm{d}r \, r \mathrm{e}^{-r^2/a} I_\nu(\varsigma r) \, I_\nu(\xi r) = \tfrac{1}{2} a \mathrm{e}^{a(\xi^2+\varsigma^2)/4} I_\nu\left(\tfrac{1}{2} a\xi\varsigma\right) \tag{2.32}$$

the result

$$\begin{aligned} g^{(1)}_{k+\beta_0}(r_b, r_a; \mathcal{E}) &= \int_0^\infty g^{(0)}_{k+\beta_0}(r_b, r; \mathcal{E}) \, g^{(0)}_{k+\beta_0}(r, r_a; \mathcal{E}) \, \mathrm{d}r \\ &= \frac{2^2}{\kappa} \int_0^\infty z h(z) \, \mathrm{d}z \end{aligned} \tag{2.33}$$

where the function $h(z)$ is defined as

$$h(z) = \frac{1}{\sinh z} \mathrm{e}^{-\kappa(r_b+r_a)\coth z} I_{2\sqrt{|k+\beta_0|^2-\alpha^2}} \left(\frac{2\kappa\sqrt{r_b r_a}}{\sinh z} \right). \tag{2.34}$$

The expression for $g^{(n)}_{k+\beta_0}(r_b, r_a; \mathcal{E})$ can be obtained by induction with respect to n, and is given by

$$g^{(n)}_{k+\beta_0}(r_b, r_a; \mathcal{E}) = \frac{2^{n+1}}{n!} \frac{1}{\kappa^n} \int_0^\infty z^n h(z) \, \mathrm{d}z. \tag{2.35}$$

Inserting the expression in equation (2.29), we obtain

$$G_k(r_b, r_a; \mathcal{E}) = \frac{2m}{\hbar} \int_0^\infty \mathrm{d}z \exp\left\{ \left(\frac{2m\beta e^2}{\hbar^2 \kappa} \right) z \right\} \frac{1}{\sinh z} \mathrm{e}^{-\kappa(r_b+r_a)\coth z} I_{2\sqrt{|k+\beta_0|^2-\alpha^2}} \left(\frac{2\kappa\sqrt{r_b r_a}}{\sinh z} \right). \tag{2.36}$$

Using the formulae (e.g. [8])

$$\int_0^\infty \mathrm{d}y\, \frac{\mathrm{e}^{2\nu y}}{\sinh y} \exp\left[-\tfrac{1}{2}t(\zeta_a+\zeta_b)\coth y\right] I_\mu\left(\frac{t\sqrt{\zeta_b\zeta_a}}{\sinh y}\right)$$
$$= \frac{\Gamma((1+\mu)/2-\nu)}{t\sqrt{\zeta_b\zeta_a}\,\Gamma(\mu+1)} W_{\nu,\mu/2}(t\zeta_b) M_{\nu,\mu/2}(t\zeta_a) \tag{2.37}$$

with the range of validity

$$\zeta_b > \zeta_a > 0$$
$$\mathrm{Re}[(1+\mu)/2-\nu] > 0$$
$$\mathrm{Re}(t) > 0 \qquad |\arg t| < \pi$$

where $M_{\mu,\nu}$ and $W_{\mu,\nu}$ are the Whittaker functions, we complete the integration and find the radial Green's function for $r_b > r_a$ in closed form,

$$G_k(r_b, r_a; E) = \frac{mc}{\sqrt{m^2c^4-E^2}} \frac{\Gamma\left(\frac{1}{2}+\sqrt{|k+\beta_0|^2-\alpha^2} - E\alpha/\sqrt{m^2c^4-E^2}\right)}{\sqrt{r_br_a}\,\Gamma\left(2\sqrt{|k+\beta_0|^2-\alpha^2}+1\right)}$$
$$\times W_{E\alpha/\sqrt{m^2c^4-E^2},\sqrt{|k+\beta_0|^2-\alpha^2}}\left(\frac{2}{\hbar c}\sqrt{m^2c^4-E^2}\,r_b\right)$$
$$\times M_{E\alpha/\sqrt{m^2c^4-E^2},\sqrt{|k+\beta_0|^2-\alpha^2}}\left(\frac{2}{\hbar c}\sqrt{m^2c^4-E^2}\,r_a\right). \tag{2.38}$$

This exact formula for the radial Green's function coincides with the earlier result obtained by one of the authors from the relativistic path integral with multivalued Levi-Civitá transformations [5]. It is worth noting that if the flux is quantized, i.e. $4\pi g = 2\pi\hbar c/e \times$ integer, $|k+\beta_0|$ is an integer and the spectrum is that of the relativistic hydrogen atom. In this case, there is no AB effect. This complete the discussions of the relativistic ABC system by the Feynman–Kac formula.

3. Concluding remarks

In the paper, a method for calculating the relativistic Green's function is given involving essentially the computation of the expectation value of moments Q^n ($Q = \int_{\lambda_a}^{\lambda_b} \mathrm{d}\lambda\, V(\boldsymbol{x})$) over the Feynman measure and summing them in accordance with the Feynman–Kac formula. As an example, the Green's function of the relativistic ABC system is given by this method. In contrast to the former treatment in [5], where the same problem must invoke the spacetime and the multivalued Levi-Civitá transformations to perform the path integral, the merit of the method used in this paper is that it involves only the explicit form of some known Green's function and an explicit path integral is avoided. We expect that the methodology presented herein might be applicable to a large number of problems.

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